Heat conduction beyond Fourier law

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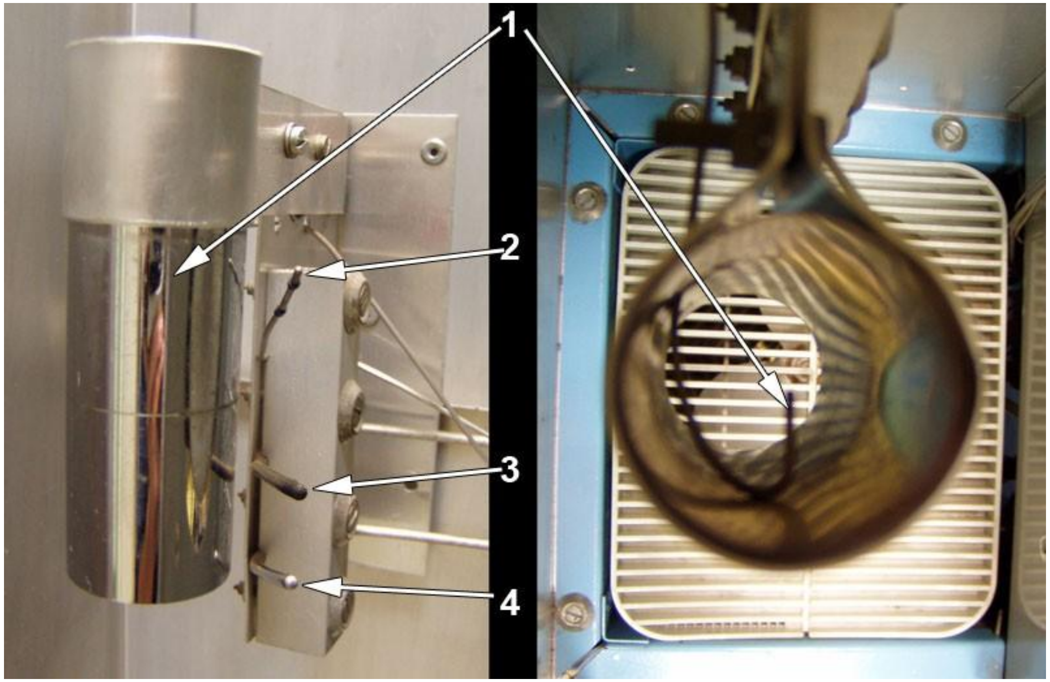
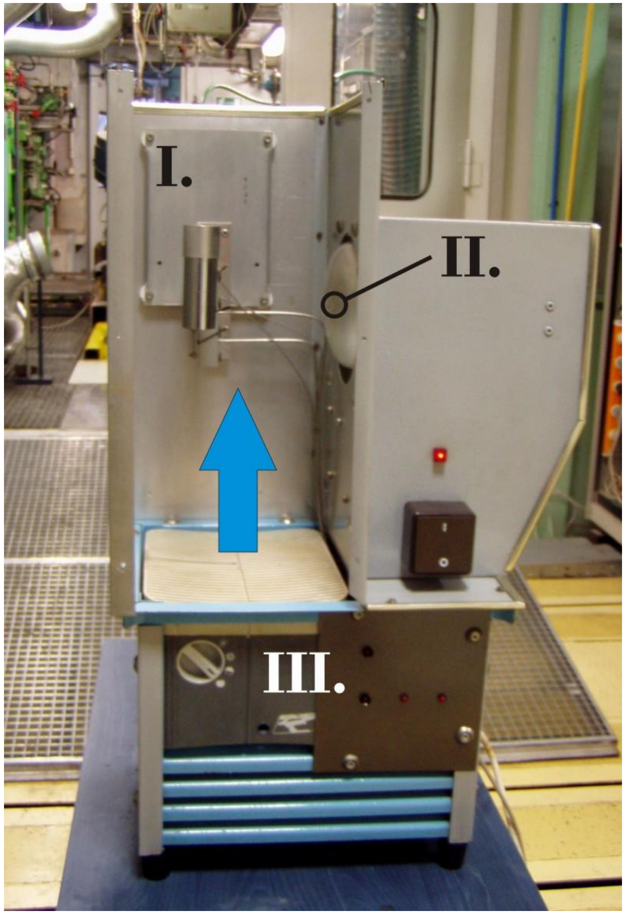
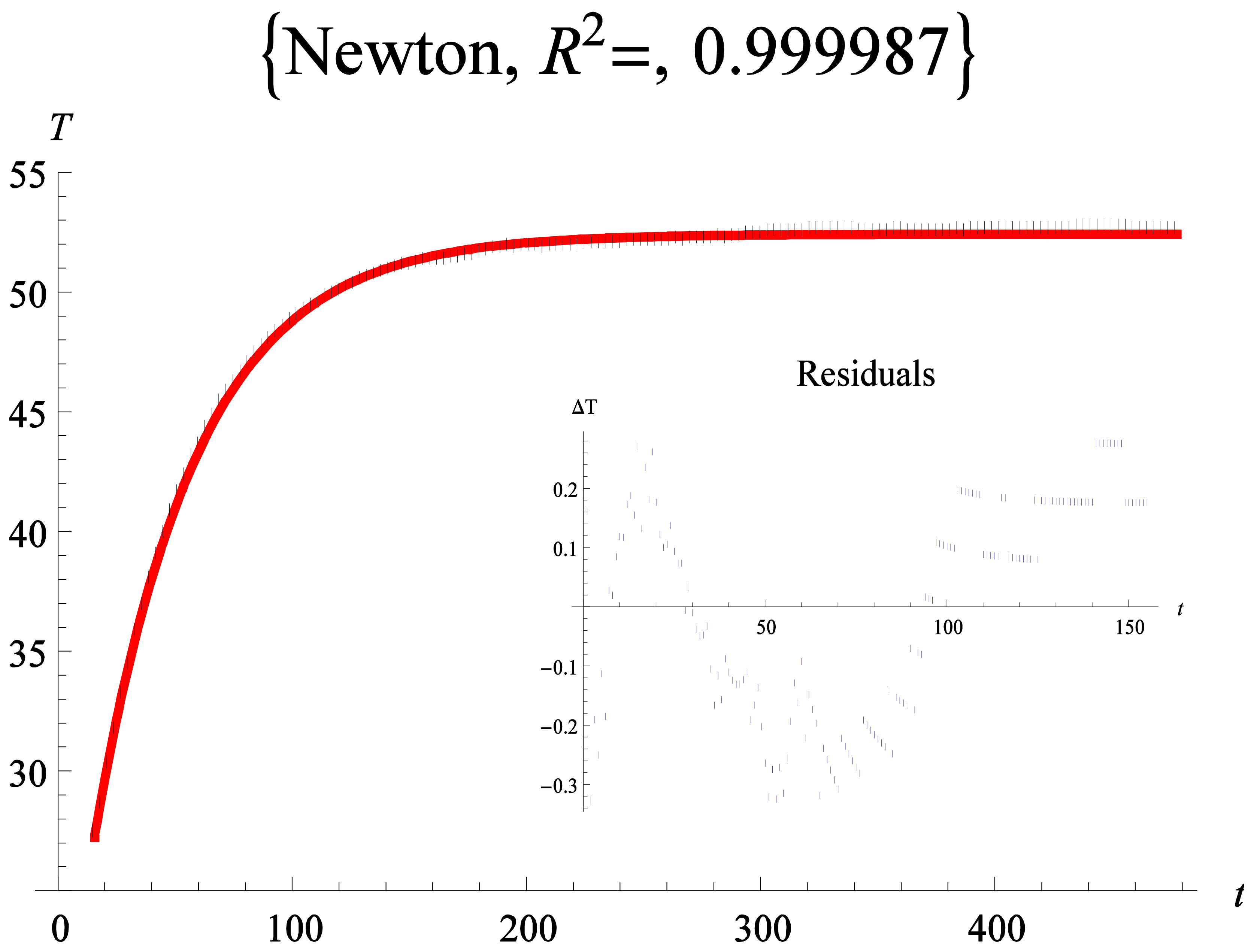
1. Introduction
2. Theories
   - non-equilibrium thermodynamics
   - solutions
3. Experiments

Heat exchange experiment





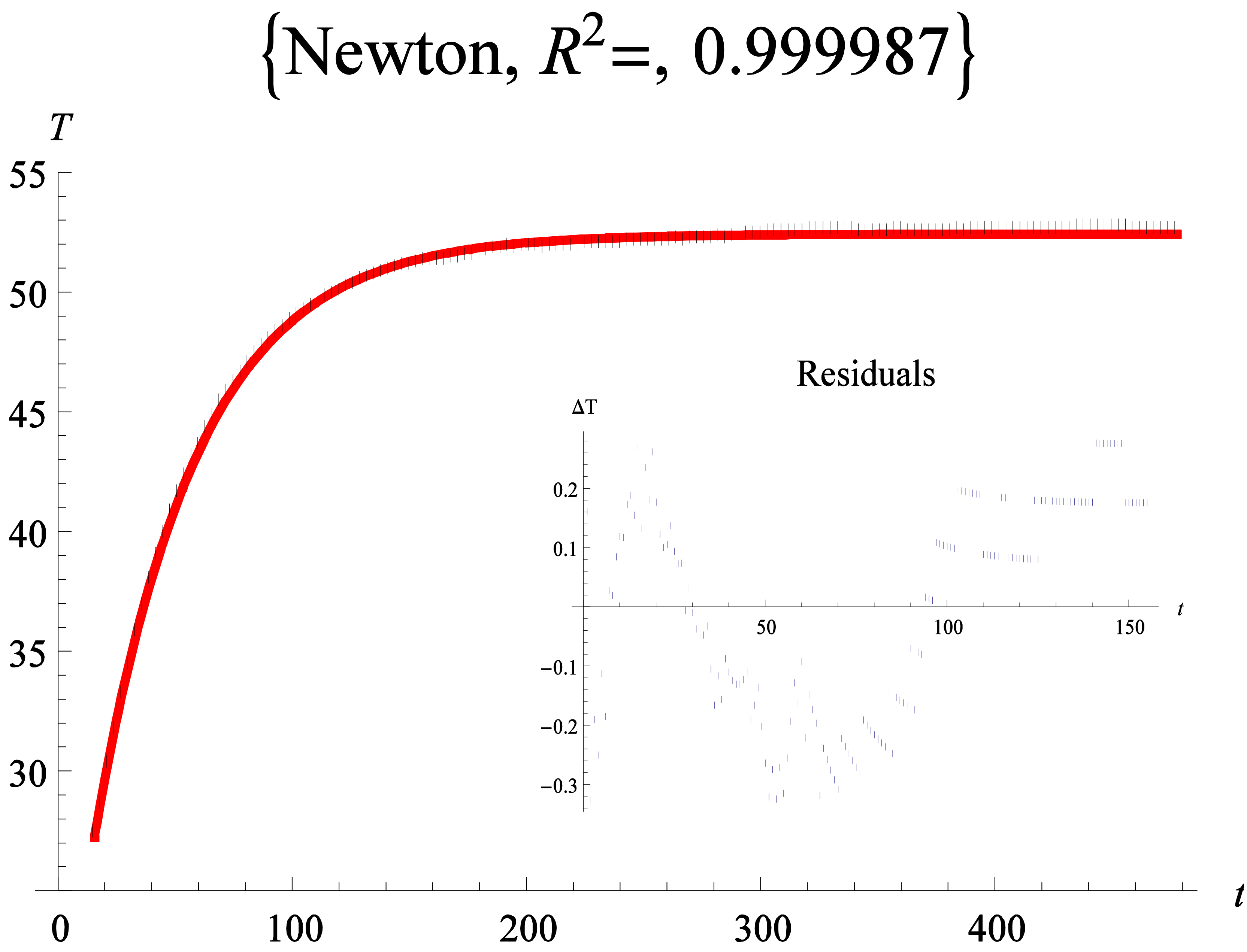
http://remotelab.energia.bme.hu/index.php?page=thermocouple_remote_desc&lang=en


{Newton, R^2=, 0.999987}
T
55
50
45
40
35
30
0
100
200
300
400
t
Residuals
ΔT
0.2
0.1
−0.1
−0.2
−0.3
50
100
150
t

Model 1 (Newton, 3 parameters):

$$\boxed{\dot{T} = -l(T - T_0)}$$

	Estimate	Std. Error
l	0.02295	0.00014
T_0	52.52	0.02
T_{ini}	27.33	0.09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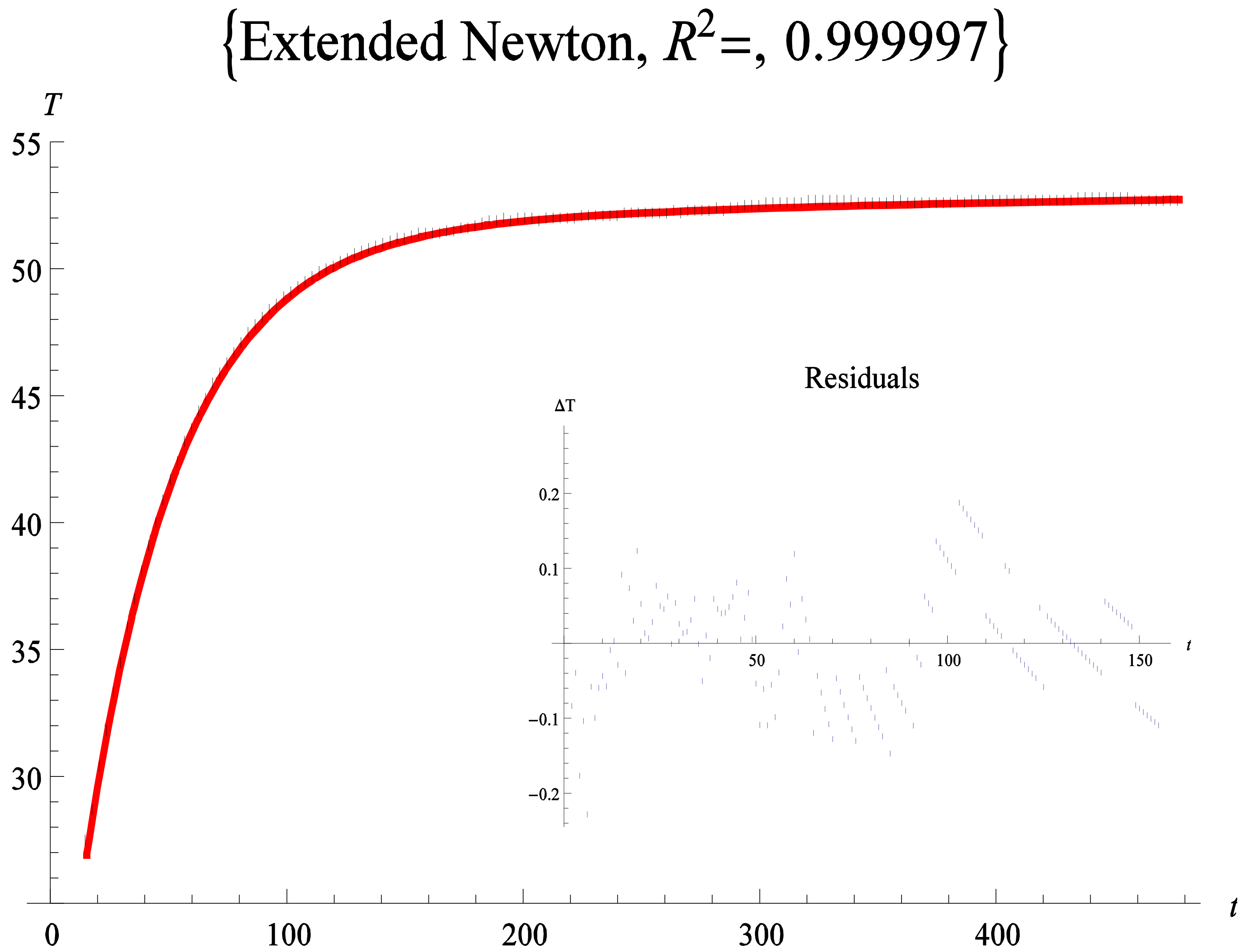
Model 2 (Extended Newton, 5 parameters):

$$\boxed{\tau\ddot{T} + \dot{T} = -l(T - T_0)}$$


{Extended Newton, R^2=, 0.999997}
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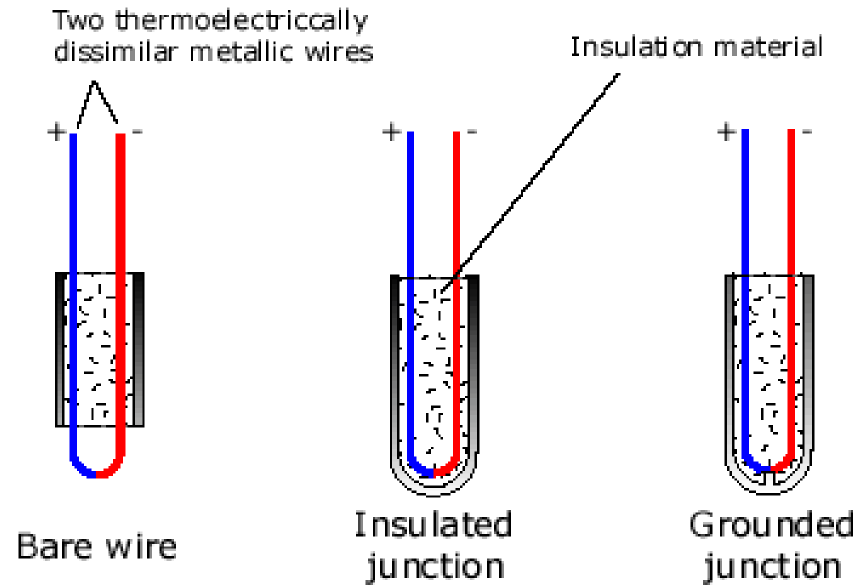
Model 2 (Extended Newton, 5 parameters):

$$\boxed{\tau\ddot{T} + \dot{T} = -l(T - T_0)}$$

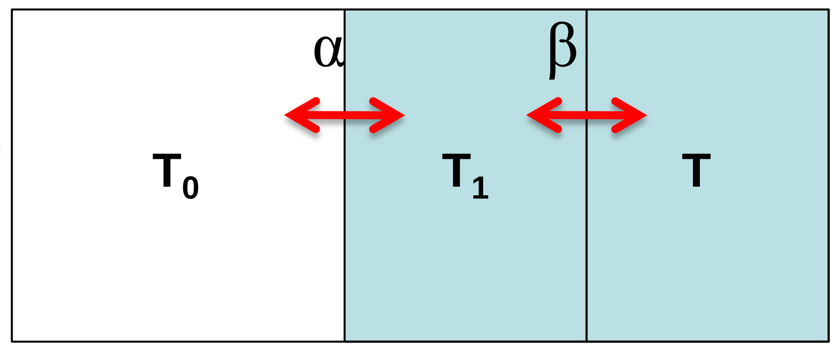
	Estimate	Std. Error
l	0.0026	0.0009
T_0	53.3	0.30
T_{ini}	26.98	0.05
τ	35.8	1.6
vT_{ini}	0.617	0.004

Why?

– two step process



$$\dot{T}_1 = -\alpha(T_1 - T_0)$$
$$\dot{T} = -\beta(T - T_1)$$



$$\frac{1}{\alpha+\beta}\ddot{T} + \dot{T} = -\frac{\alpha\beta}{\alpha+\beta}(T - T_0)$$

$$\tau\ddot{T} + \dot{T} = -l(T - T_0)$$

Oscillations in heat exchange:

– parameters

  model - values

  micro - interpretation

– macro-meso mechanism

<u>Fourier</u> – local equilibrium (Eckart, 1940)

$$\rho\dot{e} + \partial^i q^i = 0 \qquad\qquad s(e), \quad J^i = \frac{1}{T} q^j$$

$$\rho\dot{s} + \partial^i J^i \geq 0$$

Entropy production:

$$\rho\dot{s} + \partial^i J^i = \rho \frac{ds}{de}\dot{e} + \partial^i \frac{q^i}{T} = -\cancel{\frac{1}{T}\partial^i q^i} + q^i \partial^i \frac{1}{T} + \cancel{\frac{1}{T}\partial^i q^j} = q^i \partial^i \frac{1}{T} \geq 0$$

Constitutive equations (isotropy):

$$q^i = L\,\partial^i \frac{1}{T} = -\frac{L}{T^2}\,\partial^i T = -\lambda\,\partial^i T, \qquad \lambda \geq 0 \qquad \text{Fourier law}$$

Heat conduction constitutive equations

$$\rho\dot{e} + \partial^i q^i = 0$$

$$q^i = -\lambda \partial^i T,$$

Fourier (1822)

$$\tau\dot{q}^i + q^i = -\lambda \partial^i T,$$

Cattaneo (1948),
(Vernotte (1958))

$$\tau\dot{q}^i + q^i = -\lambda \partial^i T + a_1 \partial^{ij} q^j + a_2 \partial^{jj} q^i,$$

Guyer and Krumhansl (1966)

$$\tau\dot{q}^i + q^i = -\lambda \partial^i T + l \partial^i \dot{T},$$

Jeffreys type
(Joseph and Preziosi, 1989))

$$\dot{q}^i = -\lambda \partial^i T + + a_2 \partial^{jj} q^i.$$

Green-Naghdi type (1991)

$$\left(\rho c \ddot{\alpha} = k_1 \partial^{ii} \alpha + k_2 \partial^{ii} \dot{\alpha}, \qquad T = \dot{\alpha} \right)$$

there are more…

Thermodynamic approach

vectorial internal variable and current multiplier (Nyíri 1990, Ván 2001)

$$\rho\dot{e} + \partial^i q^i = 0 \qquad\qquad s\left(e - \frac{m}{2}\xi^2\right), \quad J^i = B^{ij} q^j$$

$$\rho\dot{s} + \partial^i J^i \geq 0$$

Entropy production:

$$\rho\dot{s} + \partial^i J^i = -\frac{1}{T}\partial^i q^i - \frac{m\rho}{T}\xi^i\dot{\xi}^i + \partial^i \left(B^{ij} q^j\right) =$$

$$\partial^i q^j \left(B^{ij} - \frac{1}{T}\delta^{ij}\right) + \left(\partial^i B^{ij}\right) q^j - \frac{m\rho}{T}\xi^i\dot{\xi}^i \geq 0$$

Constitutive equations (isotropy):

$$\boxed{\begin{aligned} &q^i = l_1 \partial^j B^{ji} - \hat{l}_{12}\xi^i, && \hat{l}_{12} = l_{12}\frac{\rho m}{T} \\ &\dot{\xi}^i = l_{21}\partial^j B^{ji} - \hat{l}_2 \xi^i, && \hat{l}_2 = l_2 \frac{\rho m}{T} \\ &B^{ij} - \frac{1}{T}\delta^{ij} = k_1 \partial^i q^j + k_2 \partial^j q^i + k_3 \partial^k q^k \delta^{ij}. \end{aligned}}$$

$$l_1 \geq 0, \quad l_2 \geq 0,$$
$$L = l_1 \hat{l}_2 - l_{12}\hat{l}_{21} \geq 0$$
$$k_1 \geq 0, \quad k_2 \geq 0,$$
$$k_3 \geq 0$$

$$\boxed{\tau \dot{q}^i + q^i = -\lambda_1 \partial^i T - \lambda_2 \partial^i \dot{T} + a_1 \partial^{ij} q^j + a_2 \partial^{jj} q^i + b_1 \partial^{ij} \dot{q}^j + b_2 \partial^{jj} \dot{q}^i}$$

$$\tau = \frac{1}{l_2}, \quad \lambda_1 = \frac{L}{l_2 T^2}, \quad \lambda_2 = \frac{l_1}{l_2 T^2},$$

$$a_1 = \frac{L}{l_2}(k_1 + k_3), \quad a_2 = \frac{L}{l_2} k_2, \quad b_1 = \frac{l_1}{l_2}(k_1 + k_3), \quad b_2 = \frac{l_1}{l_2} k_2,$$

1+1 D:

$$\boxed{\begin{aligned}&\rho c\dot{T}+q'=0,\\&\tau\dot{q}+q=-\lambda_1 T'-\lambda_2\dot{T}'+aq''+b\dot{q}''.\end{aligned}}$$

$$\boxed{\tau\ddot{T}+\dot{T}=\hat{\lambda}T''+\hat{a}\dot{T}''+b\ddot{T}''.}$$

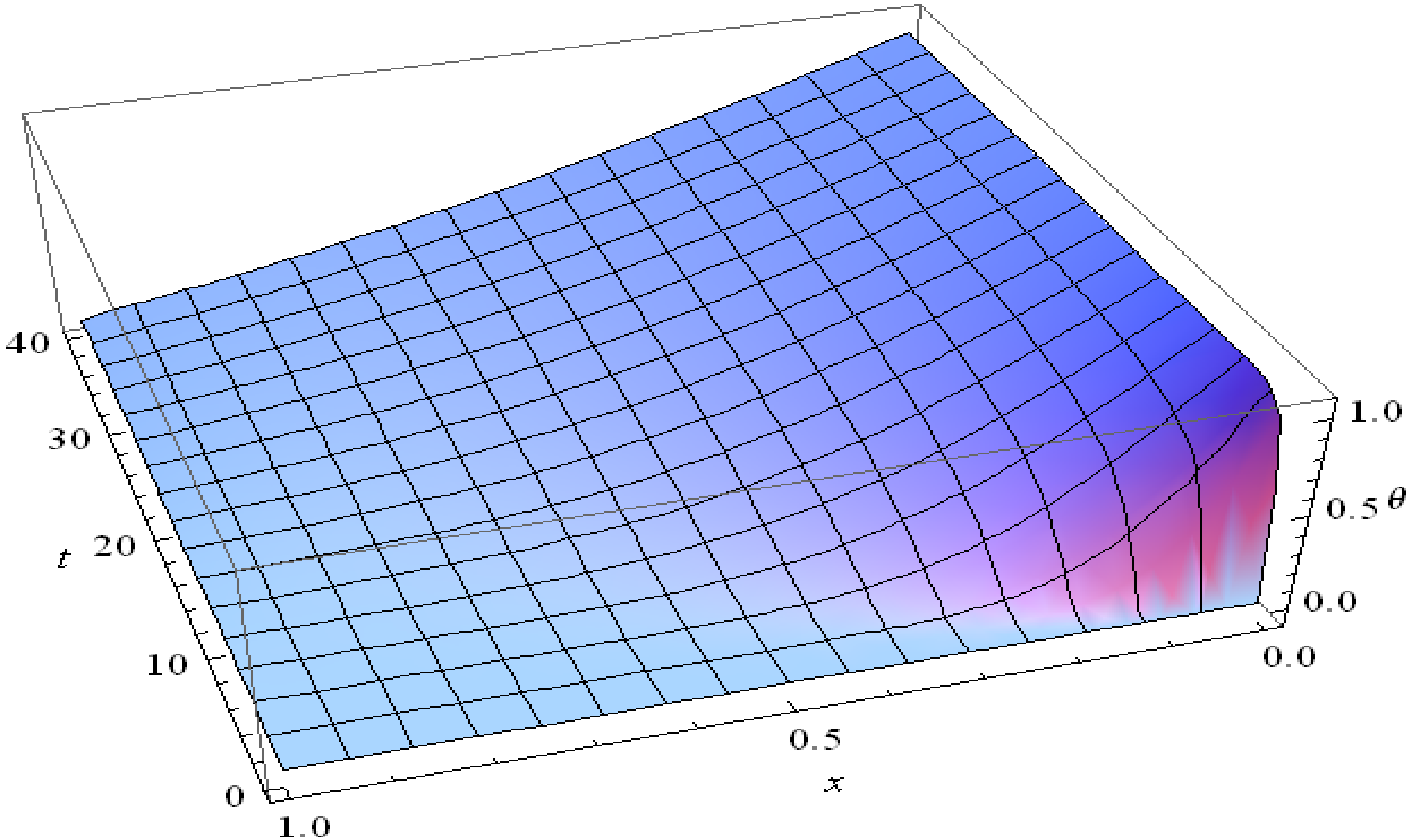
$\dot{T}=\hat{\lambda}T''$	Fourier
$\tau\ddot{T}+\dot{T}=\hat{\lambda}T''$	Cattaneo-Vernotte
$\tau\ddot{T}+\dot{T}=\hat{\lambda}T''+\hat{a}\dot{T}''$	Guyer and Krumhansl and Jeffreys type
$\tau\ddot{T}=\hat{\lambda}T''+\hat{a}\dot{T}''$	Green-Naghdi type

Theory of heat conduction

Time derivatives – internal variables
Gradients – current multipliers

Fourier

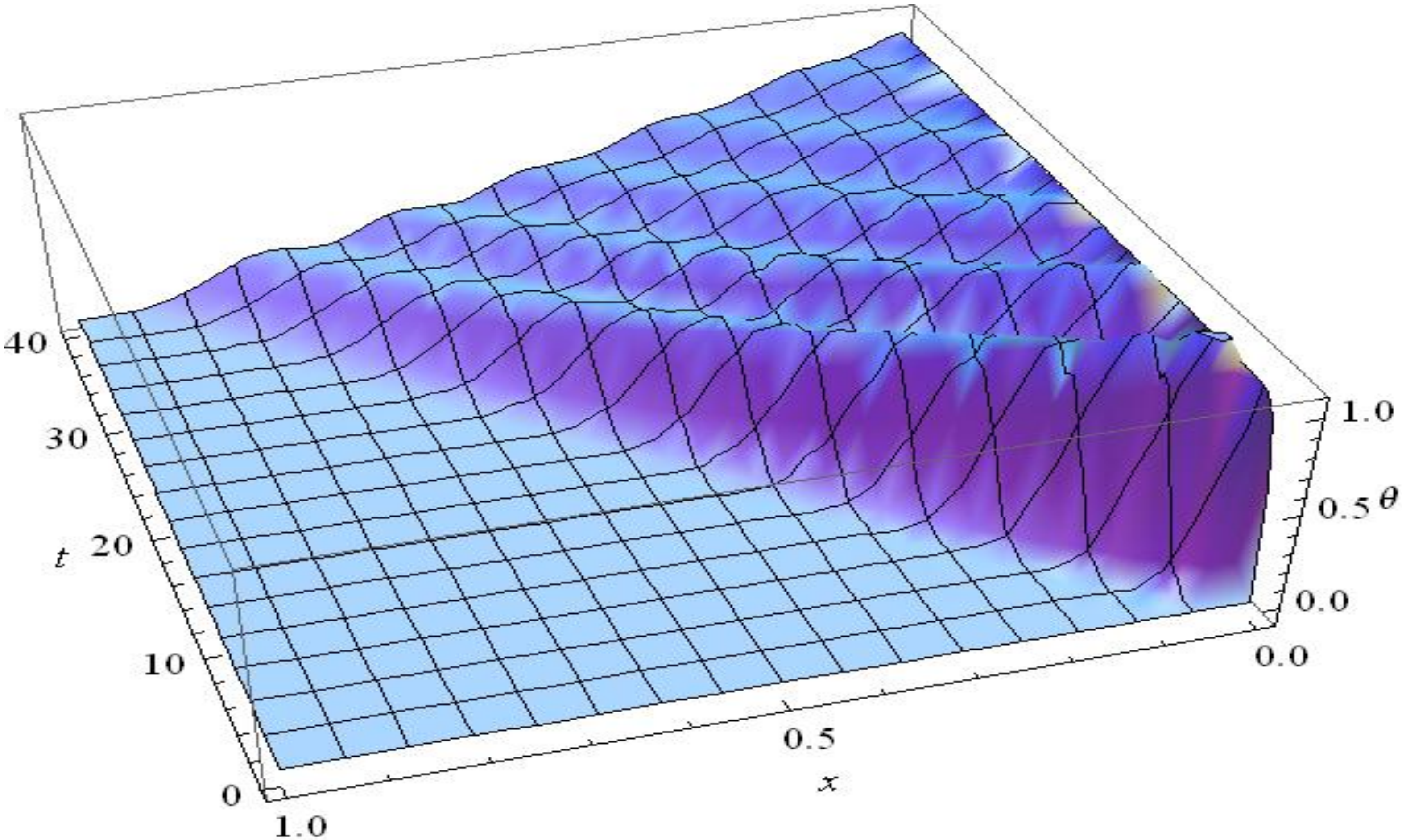
$$\boxed{\cancel{\tau\ddot{T}} + \dot{T} = \hat{\lambda}T'' + \cancel{\hat{a}\dot{T}}'' + \cancel{b\ddot{T}}''.}$$



$\tau = 0, \;\; \hat{\lambda} = 0.005, \;\; \hat{a} = 0, \;\; b = 0.00005$

Cattaneo-Vernotte

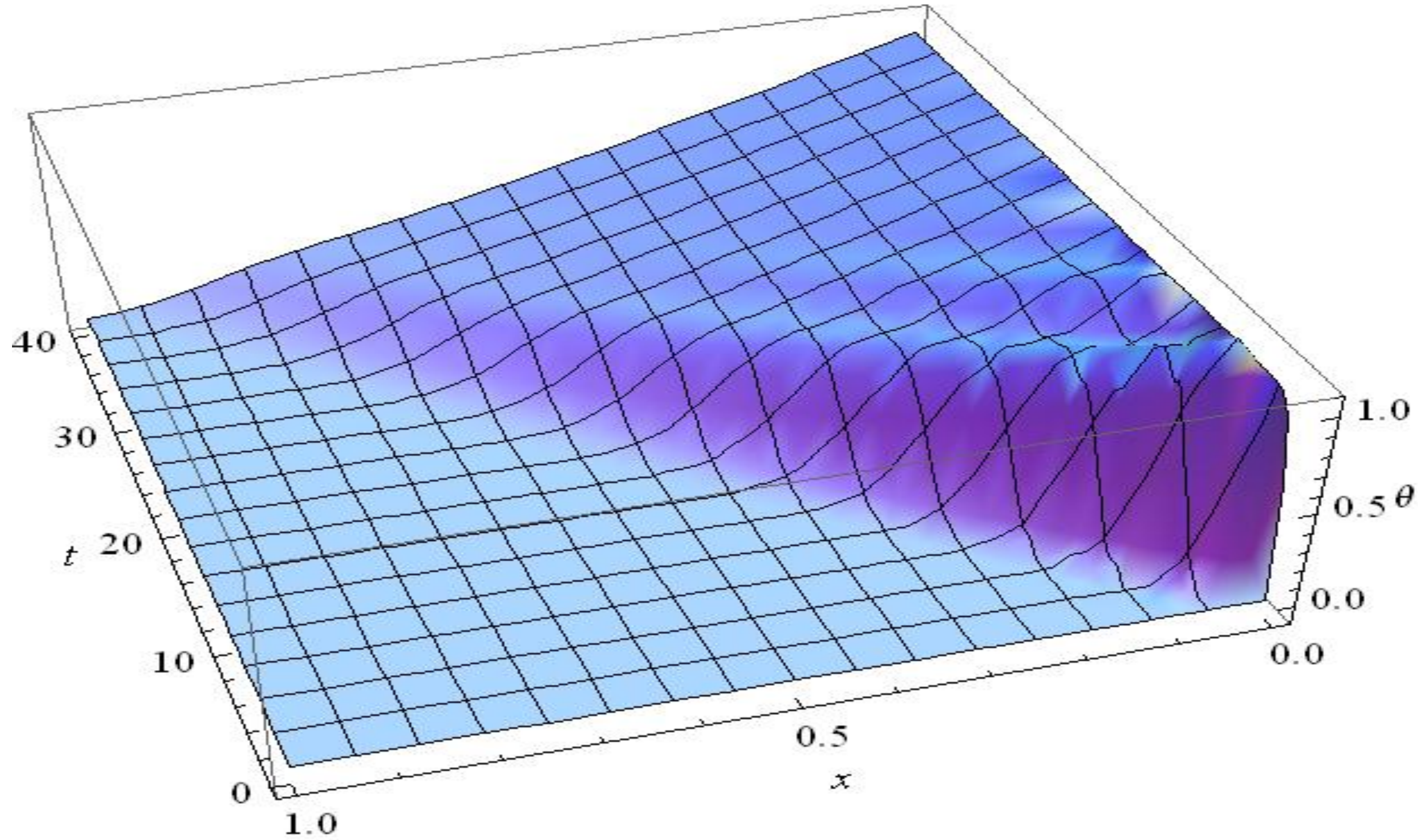
$$\boxed{\tau\ddot{T}+\dot{T}=\hat{\lambda}T''+\cancel{\hat{a}\dot{T}''}+\cancel{b\ddot{T}''}.}$$



$$\tau=10,\ \ \hat{\lambda}=0.005,\ \ \hat{a}=0,\ \ b=0.00005$$

Guyer-Krumhansl/Jeffreys type

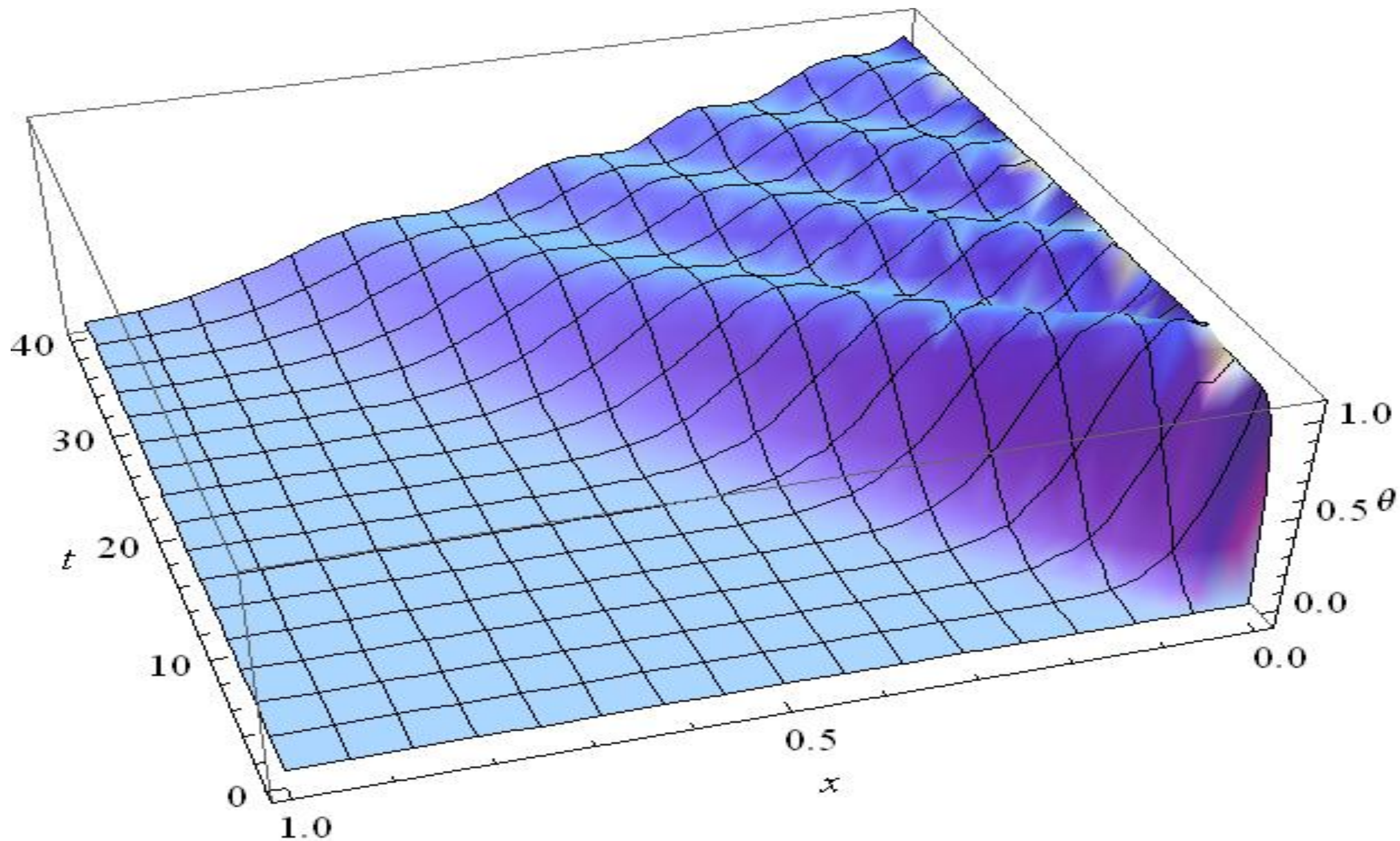
$$\tau\ddot{T} + \dot{T} = \hat{\lambda}T'' + \hat{a}\dot{T}'' + \cancel{b\ddot{T}}''.$$



$$\tau = 10,\ \hat{\lambda} = 0.005,\ \hat{a} = 0.001,\ b = 0.00005$$

Cattaneo-Vernotte + dispersion term

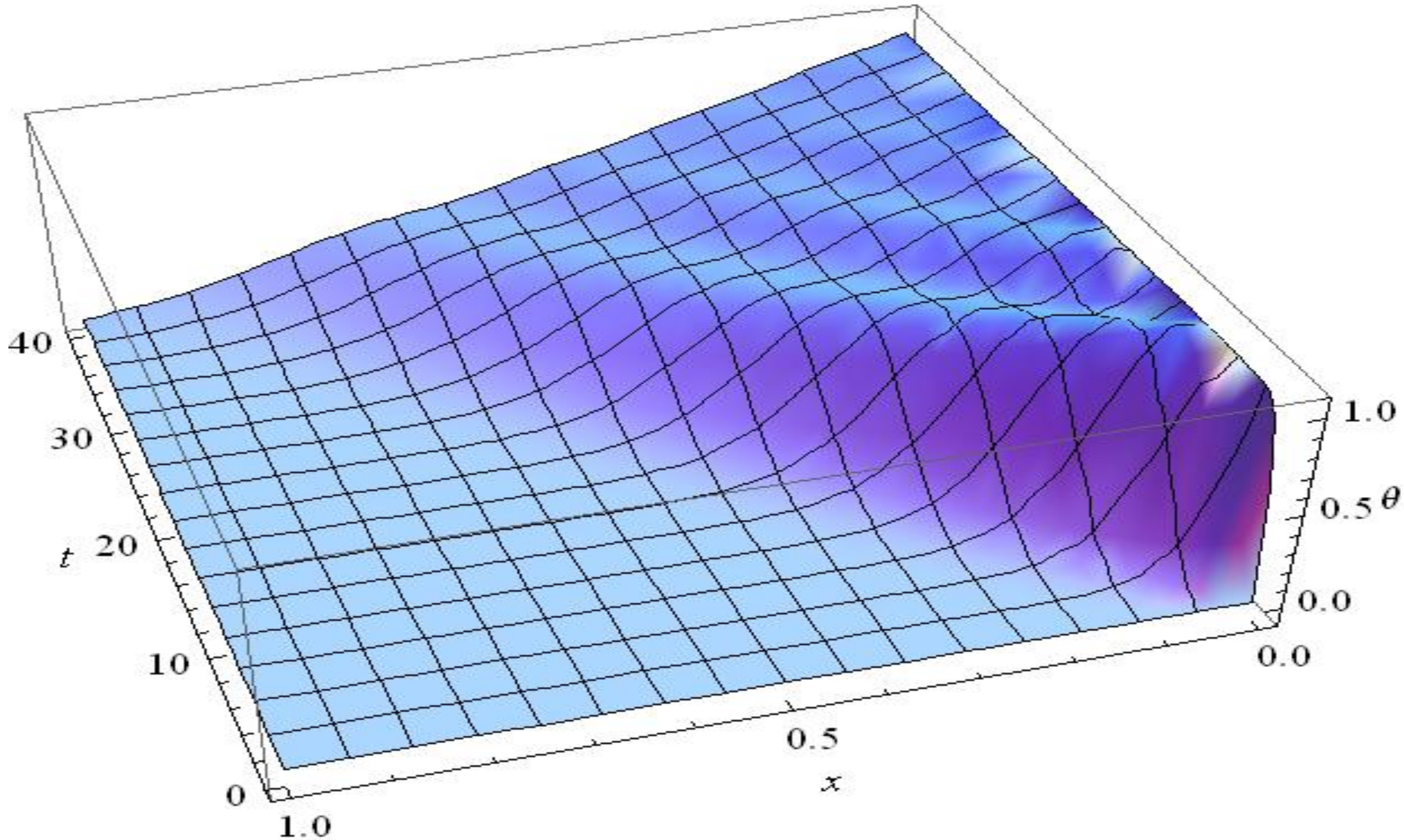
$$\tau\ddot{T} + \dot{T} = \hat{\lambda}T'' + \cancel{\hat{a}\dot{T}''} + b\ddot{T}''.$$



$\tau = 10,\ \hat{\lambda} = 0.005,\ \hat{a} = 0.00001,\ b = 0.005$

General

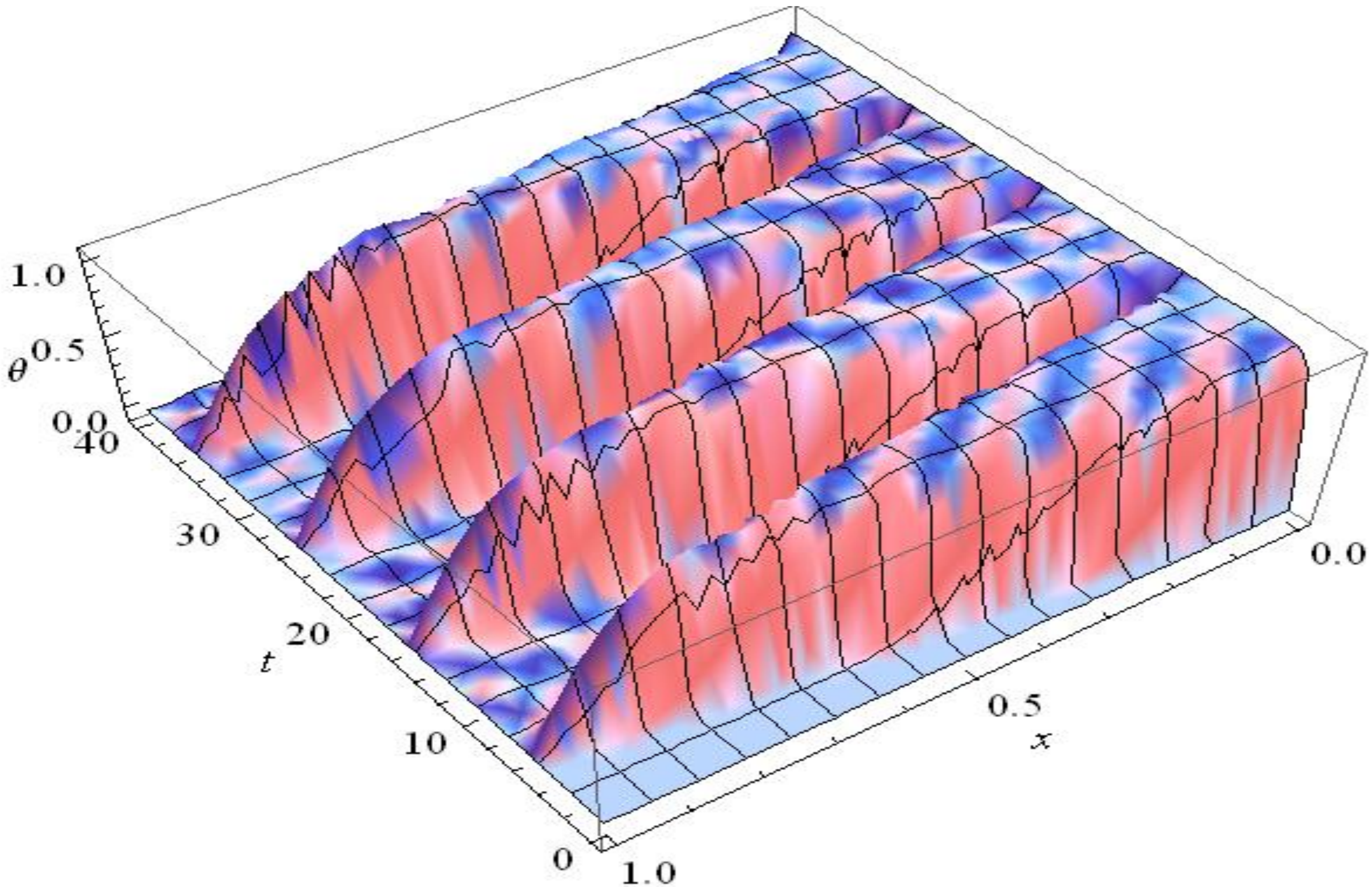
$$\boxed{\tau \ddot{T} + \dot{T} = \hat{\lambda} T'' + \hat{a} \dot{T}'' + b \ddot{T}''.}$$



$$\tau = 10,\ \ \hat{\lambda} = 0.005,\ \ \hat{a} = 0.001,\ \ b = 0.005$$

Green-Naghdi

$$\tau\ddot{T} + \cancel{\dot{T}} = \hat{\lambda}T'' + \hat{a}\dot{T}'' + b\cancel{\ddot{T}}''.$$



$$\tau = 1,\ \ \hat{\lambda} = 0.041,\ \ \hat{a} = 0.00001,\ \ b = 0.00005$$

Experiments

Homogeneous inner structure – metals

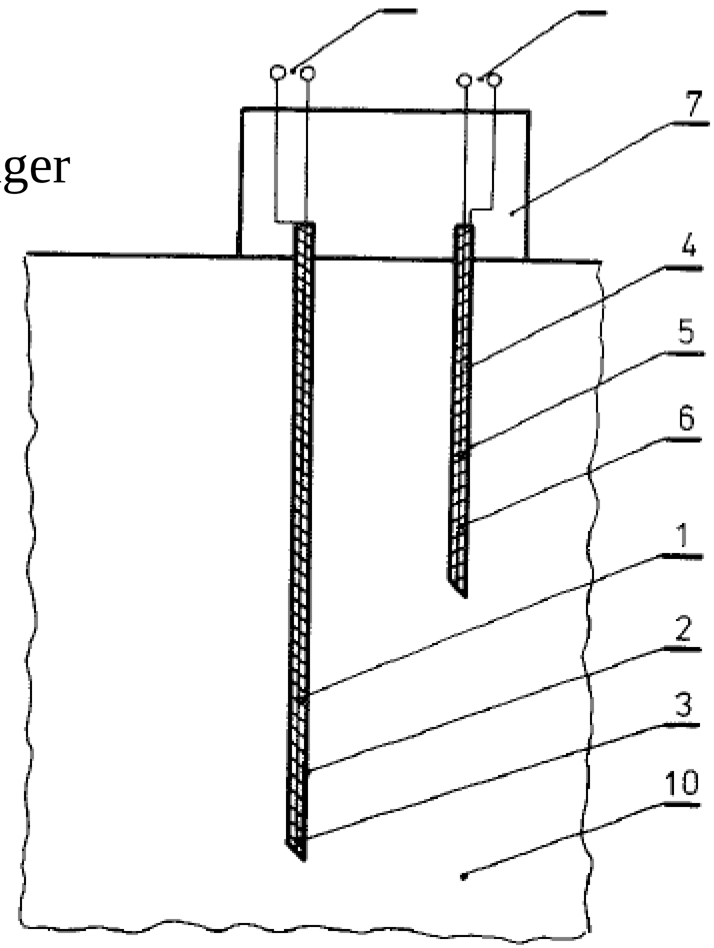
- typical relaxation times: $\tau = 10^{-13}$- 10^{-17} s
- Cattaneo-Vernotte is accepted: ballistic phonons (nano- and microtechnology?)

Inhomogeneous inner structure

- typical relaxation times: $\tau = 10^{-3}$- 100 s
- experiments are not conclusive

Kaminski, 1990

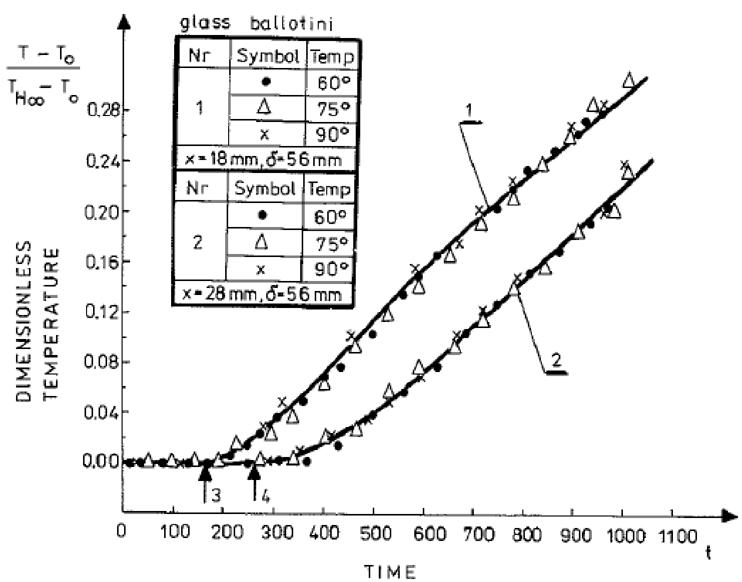
Resistance wire Thermocouple



Particulate materials:

sand, glass balottini, ion exchanger

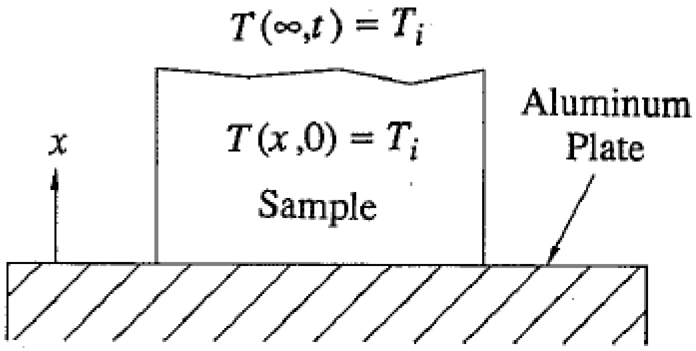
τ = 20-60 s

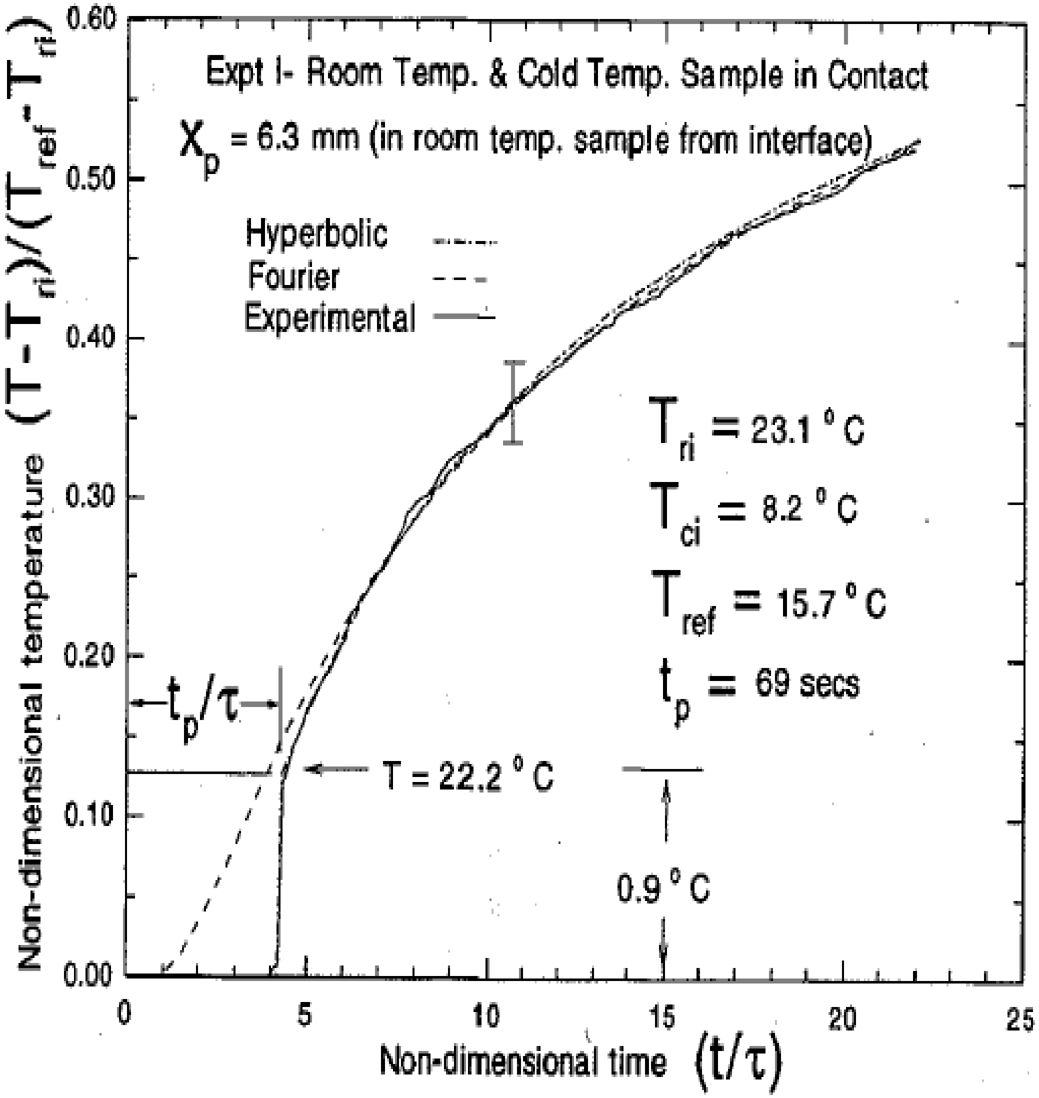


<u>Mitra-Kumar-Vedavarz-Moallemi, 1995</u>

Processed frozen meat:

τ = 20-60 s





Herwig-Beckert, 2000

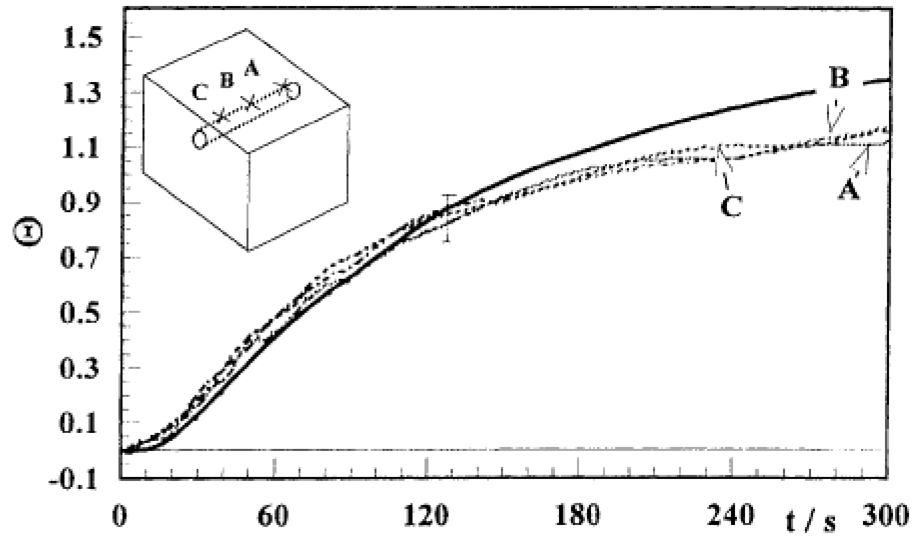
- sand, different setup
- no effect

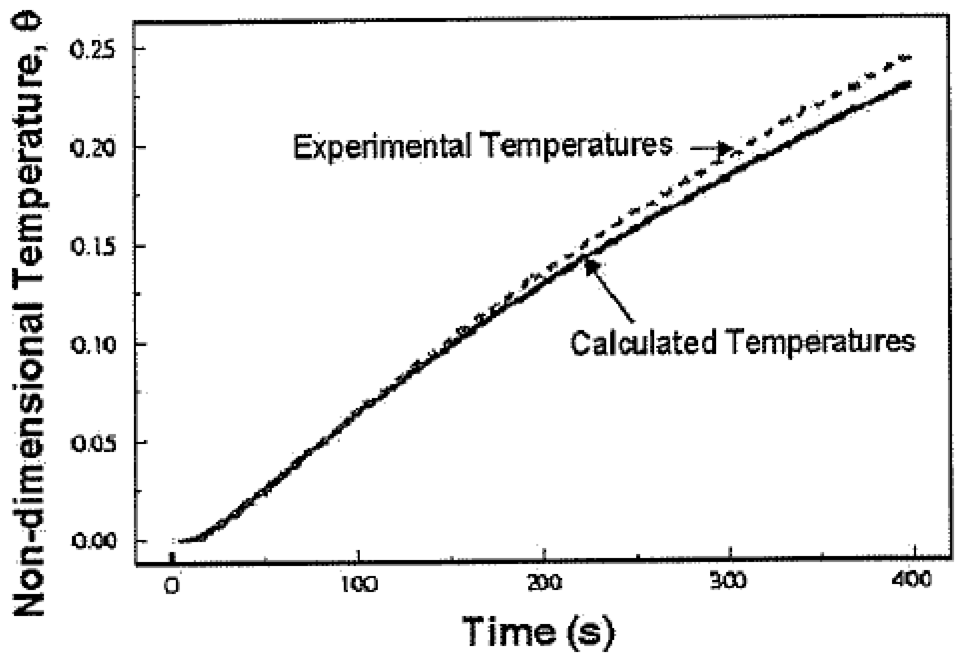
Roetzel-Putra-Das, 2003

– similar to Kaminski and Mitra et. al.
– small effect

Scott-Tilahun-Vick, 2009

– repeating Kaminski and Mitra et. al.
– no effect





Summary and conclusions

- Inertial and gradient effects
  in heat conduction
- Internal variables versus substructures
  (macro – micro)
  black box – universality
- No experiments for gradient effects
  (supressed waves?)

‘Meso’ models

a) Jeffreys-type equation – heat separation

$$\left.\begin{aligned} \hat{q}^i &= -\hat{\lambda}\partial^i T \\ \tau\dot{\tilde{q}}^i + \tilde{q}^i &= -\tilde{\lambda}\partial^i T \end{aligned}\right\} \qquad \boxed{\hat{q}^i + \tilde{q}^i = q^i}$$

1+1D:

$$\tau\dot{q} = -\tilde{q} - \tilde{\lambda}T' - \tau\hat{\lambda}\dot{T}' = -q + \hat{q} - \tilde{\lambda}T' - \tau\hat{\lambda}\dot{T}'$$

$$\boxed{\tau\dot{q} + q = -(\tilde{\lambda} + \hat{\lambda})T' - \tau\hat{\lambda}\dot{T}'}$$

Jeffreys

b) Jeffreys-type equation – dual phase lag

$$q^i(r, t + \underline{\tau_1}) = -\lambda \partial^i T(r, t + \underline{\tau_2})$$

Taylor series:

$$q^i + \tau_1 \dot{q}^i = -\lambda \partial^i T - k \tau_2 \partial^i \dot{T} \qquad \text{Jeffreys}$$

This is unacceptable.

c) Jeffreys-type equation – two steps

$$c_1 \dot{\mathrm{T}}_1 = -\partial^i q^i - g(T_1 - T_2),$$

$$q^i = -\lambda \partial^i T_1$$

$$c_2 \dot{\mathrm{T}}_2 = g(T_1 - T_2)$$

1+1D:

$$c_1 \ddot{\mathrm{T}}_1 - \lambda \dot{T}_1'' + g\dot{T}_1 = g\dot{T}_2 = \frac{g^2}{c_2}(T_1 - T_2) = \frac{g^2}{c_2} T_1 - \frac{g}{c_2}(c_1 \dot{T}_1 - \lambda \mathrm{T}_1'')$$

$$\boxed{c_1 \ddot{\mathrm{T}}_1 + g\left(1 + \frac{c_1}{c_2}\right)\dot{T}_1 = \frac{g\lambda}{c_2}\mathrm{T}_1'' + \lambda \dot{T}_1''}$$

Jeffreys

Ballistic-diffusive equation (Chen, 2001)

Boltzmann felbontás

Ballistic, analytic solution

$$\dot{f} = -\frac{f - f_0}{\tau} \qquad \begin{cases} \dot{\tilde{f}} = -\dfrac{\tilde{f}}{\tau} \\ \dot{\hat{f}} = -\dfrac{\hat{f} - f_0}{\tau} \end{cases}, \qquad f = \tilde{f} + \hat{f},$$

diffusive

1D:

$$\dot{u} + \partial_i q^i = \sigma_u, \quad u = \hat{u} + \tilde{u}, \quad q^i = \tilde{q}^i + \hat{q}^i$$

$$\boxed{\begin{aligned} \tau\dot{\tilde{u}} + \partial_i \tilde{q}^i &= -\tilde{u}, \\ \tau\dot{\hat{q}}^i + \hat{q}^i &= -\alpha \partial_i \hat{u} \end{aligned}}$$

?

Cattaneo-Vernotte equation (Gyarmati, 1977, modified)

$$\boxed{\rho\dot{e} + \partial^i q^i = 0}$$

$$\rho\dot{s} + \partial^i J^i \geq 0$$

$$s\left(e - \frac{m}{2}q^2\right), \quad J^i = \frac{1}{T}q^j$$

Entropy production:

$$\rho\dot{s} + \partial^i J^i = -\frac{1}{T}\partial^i q^i - \frac{m}{T}q^i \dot{q}^i + \partial^i \frac{q^i}{T} = q^i\left(\partial^i \frac{1}{T} - \frac{m}{T}\dot{q}^i\right) \geq 0$$

Constitutive equations (isotropy):

$$q^i = L\left(\partial^i \frac{1}{T} - \frac{m}{T}\dot{q}^i\right) \Rightarrow \boxed{\frac{mL}{T}\dot{q}^i + q^i = -\frac{L}{T^2}\partial^i T}$$

Cattaneo-Vernotte