Thermal expansion, elastic stress and finite deformation kinematics

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ABSTRACT

Kinematics + thermal expansion $\Longrightarrow$

$\Longrightarrow$ temperature gradient always causes elastic stress

Standard kinematics:

reference frame, reference t_0 , reference state/configuration:
material point $P \longleftrightarrow$ its position $\mathbf{X}$ at t_0

at t , its position is $\mathbf{x} = \mathbf{X} + \mathbf{u}_{t_0 \to t}(\mathbf{X})$

For small strains:

Cauchy strain: $\mathbf{E}^{\text{Cauchy}}_{t_0 \to t} = (\mathbf{u}_{t_0 \to t} \otimes \nabla)^{\mathsf{S}}$ $\qquad [\,\nabla \equiv \overleftarrow{\nabla} \equiv \overrightarrow{\nabla}\,]$

with the property

$$\nabla \times \mathbf{E}^{\text{Cauchy}}_{t_0 \to t} \times \nabla = \mathbf{0}$$

(Saint-Venant compatibility condition)

For nonsmall strains:

Deformation gradient: $\mathbf{F}_{t_0\to t} = \mathbf{1} + \mathbf{u}_{t_0\to t} \otimes \nabla$

Left Cauchy-Green = Finger deformation tensor:

$$\mathbf{B}_{t_0\to t} = \mathbf{F}_{t_0\to t}\mathbf{F}^{\mathsf{T}}_{t_0\to t}$$

Strain tensors: for any number n,

$$\mathbf{E}^{(n)}_{t_0\to t} = \frac{1}{n}\left(\mathbf{B}^{n/2}_{t_0\to t} - \mathbf{1}\right), \qquad \mathbf{E}^{(0)}_{t_0\to t} = \ln \mathbf{B}^{1/2}_{t_0\to t}$$

Elasticity:

$$\boldsymbol{\sigma} = 2G\,\mathbf{E}^{\text{Cauchy}}_{t_0\to t} + \left(K - \tfrac{2}{3}G\right)\left(\text{tr}\mathbf{E}^{\text{Cauchy}}_{t_0\to t}\right)\mathbf{1}\,,$$

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The recent approach to kinematics: (Fülöp–Ván, 2010)

Material point: $\mathbf{X}$ at t_0 $\longrightarrow$ $P \in \mathcal{C}$ (material manifold)

Motion: $\mathbf{u}_{t_0 \to t}(\mathbf{X})$ $\longrightarrow$ at t, P is at spacetime point $r_t(P)$

Local geometry of the motion:

$$\mathbf{F}_{t_0 \to t} = \mathbf{1} + \mathbf{u}_{t_0 \to t} \otimes \nabla \quad \longrightarrow \quad \mathbf{J}_t(P) = (r_t \otimes \widetilde{\nabla})(P)$$

Current metric (flat): **?** $\longrightarrow$ $d_t(P, Q) = \|r_t(Q) - r_t(P)\|_{\mathbf{h}}\,,$

$$\widetilde{\mathbf{h}}_t = \mathbf{J}_t^{\top} \mathbf{h} \mathbf{J}_t\,.$$

Solids: relaxed/natural metric: **?** $\longrightarrow$ $\widetilde{\mathbf{g}}$

Kinematic quantity for elasticity:

How far is the current metric from the natural metric:

Elastic shape: $\widetilde{\mathbf{A}}_t = \widetilde{\mathbf{g}}^{-1}\,\widetilde{\mathbf{h}}_t\,, \qquad \mathbf{A}_t = \mathbf{J}_t \widetilde{\mathbf{A}}_t \mathbf{J}_t^{-1}$

Deformedness: $\mathbf{D}_t = \ln \mathbf{A}_t^{1/2}$

In the natural/relaxed state: $\widetilde{\mathbf{h}}_t = \widetilde{\mathbf{g}}\,, \quad \mathbf{A}_t = \mathbf{1}\,, \quad \mathbf{D}_t = \mathbf{0}$

Elastic constitutive relation:

$$\boldsymbol{\sigma}_t = 2G\mathbf{D}_t + \left(K - \tfrac{2}{3}G\right)(\mathrm{tr}\mathbf{D}_t)\,\mathbf{1}$$

The possibility for $\widetilde{\mathbf{h}}_t = \widetilde{\mathbf{g}} \implies \widetilde{\mathbf{g}}$ also flat: its $\mathbf{R}^{\mathrm{Ricci}} = \mathbf{0}$:

$$
\begin{aligned}
&\mathrm{tr}_{1,5;3,4}\left[\mathbf{A}^{-1}\otimes\mathbf{A}^{-1}\left(\mathbf{A}\otimes\nabla\right)\left(\mathbf{A}\otimes\nabla\right)\right]+\\
&+2\mathbf{A}^{-1}\,\mathrm{tr}_{1,2}\left[\mathbf{A}\left(\nabla\otimes\nabla\otimes\mathbf{A}\right)\right]\mathbf{A}^{-1}+\\
&+2\mathrm{tr}_{2,3}\left[\mathbf{A}^{-1}\otimes\left(\nabla\cdot\mathbf{A}\right)\left(\nabla\otimes\mathbf{A}\right)\right]\mathbf{A}^{-1}+\\
&+2\mathrm{tr}_{1,2}\left[\mathbf{A}^{-1}\left(\mathbf{A}\otimes\nabla\otimes\nabla\right)\right]-2\mathbf{A}^{-1}\,\mathrm{tr}_{2,4}\left[\left(\mathbf{A}\otimes\nabla\otimes\nabla\right)\right]+\\
&+2\mathrm{tr}_{1,4;3,5}\left[\mathbf{A}^{-1}\otimes\mathbf{A}^{-1}\left(\mathbf{A}\otimes\nabla\right)\left(\mathbf{A}\otimes\nabla\right)\right]+\\
&+\mathrm{tr}_{1,2;3,5}\left[\mathbf{A}^{-1}\left(\mathbf{A}\otimes\nabla\right)\left(\mathbf{A}\otimes\nabla\right)\otimes\mathbf{A}^{-1}\right]-\\
&-2\mathbf{A}^{-1}\,\mathrm{tr}_{2,5;3,6}\left[\left(\mathbf{A}\otimes\nabla\right)\mathbf{A}\left(\nabla\otimes\mathbf{A}\right)\otimes\mathbf{A}^{-1}\right]\mathbf{A}^{-1}-\\
&-2\left[\nabla\otimes\left(\nabla\cdot\mathbf{A}\right)\right]\mathbf{A}^{-1}+2\mathrm{tr}_{2,4}\left[\left(\nabla\otimes\mathbf{A}\right)\mathbf{A}^{-1}\left(\mathbf{A}\otimes\nabla\right)\right]\mathbf{A}^{-1}-\\
&-3\mathrm{tr}_{2,6;3,5}\left[\left(\nabla\otimes\mathbf{A}\right)\mathbf{A}^{-1}\left(\mathbf{A}\otimes\nabla\right)\otimes\mathbf{A}^{-1}\right]-\\
&-\mathrm{tr}_{3,4;2,5}\left[\mathbf{A}^{-1}\otimes\mathbf{A}^{-1}\left(\mathbf{A}\otimes\nabla\right)\mathbf{A}\left(\nabla\otimes\mathbf{A}\right)\right]\mathbf{A}^{-1}-\\
&-2\mathbf{A}^{-1}\,\mathrm{tr}_{2,4;3,5}\left[\left(\mathbf{A}\otimes\nabla\right)\otimes\left(\nabla\otimes\mathbf{A}\right)\right]\mathbf{A}^{-1}=\mathbf{0}
\end{aligned}
$$

Small deformedness: $\mathbf{D} \ll \mathbf{1}$:

$$\nabla \times \mathbf{D} \times \nabla + (\text{higher order terms}) = \mathbf{0}$$

(recall $\nabla \times \mathbf{E}^{\text{Cauchy}}_{t_0 \to t} \times \nabla = \mathbf{0}$)

Connection to the conventional kinematics:

$$\mathbf{F}_{t_0 \to t} = \mathbf{J}_t \mathbf{J}_{t_0}^{-1}$$

The initial state is not relaxed, in general;

$$\mathbf{A}_t = \mathbf{F}_{t_0 \to t}\, \mathbf{A}_{t_0}\, \mathbf{F}^{\mathsf{T}}_{t_0 \to t} \qquad \text{(compare to } \mathbf{B}_{t_0 \to t} = \mathbf{F}_{t_0 \to t} \mathbf{F}^{\mathsf{T}}_{t_0 \to t} \text{)}$$

Thermal expansion:

$$\widetilde{\mathbf{g}}^{(T)} = \Lambda^{(T_0,T)}\, \widetilde{\mathbf{g}}^{(T_0)}$$

$\Lambda^{(T_0,T)}$: thermal scaling factor

If $T \approx T_0$:

$$\Lambda^{(T_0,T)} \approx \left[1 + \alpha^{(T_0)}(T - T_0)\right]^2 \approx 1 + 2\alpha^{(T_0)}(T - T_0)$$

$\alpha^{(T_0)}$: linear thermal expansion coefficient at T_0

If T_0 at t_0 , T at t :

$$\mathbf{A}_t = \frac{1}{\Lambda^{(T_0,T)}}\, \mathbf{F}_{t_0 \to t}\, \mathbf{A}_{t_0}\, \mathbf{F}^{\mathsf{T}}_{t_0 \to t}$$

If $T \approx T_0$, $\mathbf{D}_t \ll \mathbf{1}$, $\mathbf{F}_{t_0 \to t} \approx \mathbf{1}$:

$$\mathbf{D}_t \approx \tfrac{1}{2}\left(\mathbf{A}_t - \mathbf{1}\right) \approx \tfrac{1}{2}\Big(\mathbf{F}_{t_0 \to t}\mathbf{F}^{\mathsf{T}}_{t_0 \to t} - \mathbf{1}\Big) - \tfrac{1}{2}\Big(\Lambda^{(T_0,T)} - 1\Big)\mathbf{1}$$

$$\approx \mathbf{E}^{\mathrm{Cauchy}}_{t_0 \to t} - \alpha^{(T_0)}(T - T_0)\,\mathbf{1}\,,$$

and $\boldsymbol{\sigma}_t = 2G\mathbf{D}_t + \left(K - \tfrac{2}{3}G\right)(\mathrm{tr}\mathbf{D}_t)\,\mathbf{1}$ expands to

$$\boldsymbol{\sigma}_t \approx 2G\mathbf{E}^{\mathrm{Cauchy}}_{t_0 \to t} + \left(K - \tfrac{2}{3}G\right)\mathbf{E}^{\mathrm{Cauchy}}_{t_0 \to t} - 3K\alpha^{(T_0)}(T - T_0)\mathbf{1}\ :$$

the Duhamel-Neumann expression

A general result: $\Lambda^{(T_0,T)}$ makes $\widetilde{\mathbf{g}}$ space-and-time dependent

$$\Longrightarrow \quad \widetilde{\mathbf{g}} \text{ is no longer flat}$$

$$\Longrightarrow \quad \widetilde{\mathbf{g}} \text{ cannot be equal to any } \widetilde{\mathbf{h}}_t$$

$$
\begin{aligned}
\mathbf{R}^{\text{Ricci}} = \mathbf{R}^{\text{Ricci}}\Big|_{\Lambda=1} + \\
&+ \frac{1}{4\Lambda^2}\Big\{3(\nabla\Lambda)\otimes(\nabla\Lambda) - 2\Lambda(\nabla\otimes\nabla\Lambda) - \Lambda\mathrm{tr}_{2,3}\big[(\nabla\otimes\mathbf{A})(\nabla\Lambda)\big]\mathbf{A}^{-1} \\
&- \Lambda\mathbf{A}^{-1}\mathrm{tr}_{2,4}\big[(\mathbf{A}\otimes\nabla)\otimes(\nabla\Lambda)\big] - 2\Lambda\big[(\nabla\Lambda)(\mathbf{A}\cdot\nabla)\big]\mathbf{A}^{-1} \\
&- 2\Lambda\mathrm{tr}\big[(\nabla\otimes\nabla\Lambda)\mathbf{A}\big]\mathbf{A}^{-1} + \Lambda\mathbf{A}^{-1}\big[(\nabla\Lambda)\mathbf{A}(\nabla\otimes\mathbf{A})\big]\mathbf{A}^{-1} \\
&+ \big[(\nabla\Lambda)\mathbf{A}(\nabla\Lambda)\big]\mathbf{A}^{-1} + \Lambda(\nabla\Lambda)\mathbf{A}\mathrm{tr}_{2,3}\big[(\nabla\otimes\mathbf{A})\mathbf{A}^{-1}\big]\mathbf{A}^{-1}\Big\}
\end{aligned}
$$

$\mathbf{A} = \mathbf{1}$: $\frac{1}{4\Lambda^2}\Big\{3(\nabla\Lambda)\otimes(\nabla\Lambda) - 2\Lambda(\nabla\otimes\nabla\Lambda) + \big[(\nabla\Lambda)(\nabla\Lambda) - 2\Lambda(\triangle\Lambda)\big]\mathbf{1}\Big\}$

$$\nabla T \Longrightarrow \nabla\Lambda \Longrightarrow \mathbf{D} \Longrightarrow \boldsymbol{\sigma}$$

A general result: $\Lambda^{(T_0,T)}$ makes $\widetilde{\mathbf{g}}$ space-and-time dependent

$\Longrightarrow$ $\widetilde{\mathbf{g}}$ is no longer flat

$\Longrightarrow$ $\widetilde{\mathbf{g}}$ cannot be equal to any $\widetilde{\mathbf{h}}_t$

$$\mathbf{R}^{\text{Ricci}} = \mathbf{R}^{\text{Ricci}}\Big|_{\Lambda=1} +$$

$$+ \frac{1}{4\Lambda^2}\Big\{3(\nabla\Lambda)\otimes(\nabla\Lambda) - 2\Lambda(\nabla\otimes\nabla\Lambda) - \Lambda\mathrm{tr}_{2,3}\big[(\nabla\otimes\mathbf{A})(\nabla\Lambda)\big]\mathbf{A}^{-1}$$

$$- \Lambda\mathbf{A}^{-1}\mathrm{tr}_{2,4}\big[(\mathbf{A}\otimes\nabla)\otimes(\nabla\Lambda)\big] - 2\Lambda\big[(\nabla\Lambda)(\mathbf{A}\cdot\nabla)\big]\mathbf{A}^{-1}$$

$$- 2\Lambda\mathrm{tr}\big[(\nabla\otimes\nabla\Lambda)\mathbf{A}\big]\mathbf{A}^{-1} + \Lambda\mathbf{A}^{-1}\big[(\nabla\Lambda)\mathbf{A}(\nabla\otimes\mathbf{A})\big]\mathbf{A}^{-1}$$

$$+ \big[(\nabla\Lambda)\mathbf{A}(\nabla\Lambda)\big]\mathbf{A}^{-1} + \Lambda(\nabla\Lambda)\mathbf{A}\mathrm{tr}_{2,3}\big[(\nabla\otimes\mathbf{A})\mathbf{A}^{-1}\big]\mathbf{A}^{-1}\Big\}$$

$\mathbf{A} = \mathbf{1}$: $\frac{1}{4\Lambda^2}\Big\{3(\nabla\Lambda)\otimes(\nabla\Lambda) - 2\Lambda(\nabla\otimes\nabla\Lambda) + \big[(\nabla\Lambda)(\nabla\Lambda) - 2\Lambda(\triangle\Lambda)\big]\mathbf{1}\Big\}$

$$\nabla T \Longrightarrow \nabla\Lambda \Longrightarrow \mathbf{D} \Longrightarrow \boldsymbol{\sigma}$$

THANK YOU FOR YOUR ATTENTION!